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ABSTRACT

This paper contains the comments of a discussant reviewing the paper, "The Role of Visual Representations in the Learning of Mathematics" (Abraham Arcavi). A research-based evaluation is made that focuses on the ideas of stochastics and the epistemological triangle. (Contains 28 references.) (DDR)

DISCUSSANT'S COMMENTS: ON THE ROLE OF VISUAL REPRESENTATIONS IN THE LEARNING OF MATHEMATICS

Sunday A. Ajose
East Carolina University, USA
ajoses@mail.ecu.edu

Making visual representations of things is a natural cognitive activity, which is valued for good reasons. For example, visual images can facilitate the recall of facts and events; they can also be crucial in the search for solution(s) to mathematical problems (Polya, 1945). Not only that, anyone who has studied mathematics can, in all likelihood, recall an instance or two where a visual clue made all the difference in his/her learning of a mathematical concept or procedure. In spite of these advantages, "visual representation remains a second-class citizen in both the theory and practice of mathematics" (Barwise and Etchemendy, 1991). Most mathematicians just distrust visual thinking, and look down on proofs that make heavy use of that form of thinking. Professor Arcavi seems intent on changing these old attitudes.

In his paper, Professor Arcavi makes a strong case in support of a small but growing group of mathematicians who want to "make visual reasoning an acceptable practice of mathematics, alongside, and in combination with algebraic reasoning."

To begin his enlightening exploration of visualization and its role in the learning of mathematics, Arcavi defines visualization as "the ability, the process and the product of creation, interpretation, and use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings." He then identifies three roles which visualization may play in the learning process. Visualization, he asserts, can serve as (a) support and illustration of essentially symbolic results (and possibly providing a proof in its own right), (b) a possible way of resolving conflict between (correct) symbolic solutions and (incorrect) intuitions, and (c) a way to help us engage with and recover conceptual underpinnings which may be easily bypassed by formal solutions" to problems. To defend these claims, Professor Arcavi uses engaging problems to show real situations where visualization actually plays these roles. I find these illustrations quite compelling; they make a case with which few, if any, in this assembly would disagree.

and means and conditions to communicate ideas, are to be considered in respect to a specific mathematical content. This mathematical model is a system of abstract entities and relations logically and deductively stated, from axiomatic grounds, as definitions, theorems, corollaries, propositions. The terms we use to refer to the mathematical content in relation to the individual, such as *notions*, *ideas*, *concepts*, are not deprived of ambiguity. Often they are indistinctly used. For instance, *concept* is defined in the dictionary as something *conceived* in the mind, as a thought, a notion; as an abstract or generic idea generalized from particular instances (Merriam-Webster's Dictionary). However, in order to take into account different levels of abstraction in the individual's intellectual activity, we refer to notion, then idea and finally concept.

The term concept has been borrowed from philosophy, in particular from the analytic school of philosophy, where concept is a logical entity. From a social perspective regarding knowledge and its constitution in the individual, Sfard (1996) quotes Foucault to characterise *concept* in discursive terms, as '...a virtual entity "constituted by all that was said in all the statements that named it, divided it up, described it, explained it, traced its developments, indicated its various correlations, judged it ..."' (p. 403). In addition to the conditions that our sensorial system imposes on the way we perceive our reality (Schmidt, 1996, pp. 386-388), and to our daily experience (in a social community), by means of education we have a view of the world structured from projecting over it (the world) our concepts. Once certain concepts are introduced in a determined way, we only can use them by following the profiles that reality adopts by projecting over reality those concepts (Mosterin, 1964, pp. 11-39); that is, we consider reality according to the conceptual schema we use for that consideration. A concept is a rule that may be applied to decide if a particular object falls into a certain class. Concept formation refers to the process by which one learns to sort one's specific experiences into general rules or classes; whereas conceptual thinking refers to one's subjective manipulations (that is, to treat, relate or operate with) those abstract classes.

Fundamental ideas of stochastics

As a general orientation for our project, we have regarded ideas of probability and statistics as fundamental in the sense in which Heitele has spelled them out in his proposition for the curriculum of stochastics (1975). That is, as those ideas that provide the individual with an explanatory model of the (random) situation with which she or he is concerned, with no changes in essence but in their linguistic presentation and sophistication at the different levels of his or her conceptual development. We could add to this characterisation the meaning of *model* as "a system of postulates, data, and inferences presented as [or that provide the individual with] a mathematical description of an entity or state of affairs", after The Merriam-Webster's Dictionary. More specifically, Heitele proposed as fundamental ideas for the curriculum of stochastics the following: norming our beliefs (in the mathematical sense of norm), sample space, addition of probabilities, independence and the product of probabilities, equiprobability and symmetry, combinatorics, random variable, the law of large numbers, sample, urn model and simulation. It is by posing in the teaching of probability rich situations from whose study several interrelations among these ideas could be laid out, that chance, and probability, can be put into focus (Ojeda, 1994; González, 1995; Alquicira, 1998).

However, this general guide is not deprived of the main drawback one has when facing chance, since a duality lies at the very meaning of probability. The fact that the meaning of probability has so much attracted the attention of philosophers and scientists of all times (e. g. Hacking, 1975; Krüger *et al.*, 1987) suggests that probability requires a different way of thinking from the one needed for other mathematical concepts. Lack of instruction in this subject matter would result in an incomplete background to face a wide range of world situations. Freudenthal pointed out that "the usual mistakes in this field differ greatly from mistakes in mathematical techniques. Those who never had the opportunity of making these mistakes, also did not either get the opportunity to *unlearn* them" (p. 587).

Yet this complexity does not imply that education in probability be beyond the scope of children before the age of preparatory or university level. On the contrary, delay of instruction in this discipline may result in the rooting of misconceptions (Fischbein, 1975). Even pre-school children and children aged 6-8 years old can be posed activities involving chance (Limón, 1995; Gurrola, 1998), although appropriate teacher training should be required (López, 1998).

Freudenthal expressed the importance of probability and statistics in the mathematics educational task (1973, p. 581) stating that "probability provides the best opportunity to show students how to mathematize, how to apply mathematics - not only the best, but perhaps even the next and last opportunity after elementary arithmetic ..." (p. 592), as this topic is the privileged field of mathematics applications. Still research in primary education has offered evidence that a correct performance with fractions does not imply a correct performance in probability, and that correct insight in probability may occur without a correct handling of fractions (Perrusquía, 1998).

Freudenthal recognized the difficulty involved in the concept of probability, and expressed this by quoting Poincaré (1896) at the beginning of his discussion:

> *Calcule des Probabilités: Première Leçon. 1. L'on ne peut guère donner une définition satisfaisante de la Probabilité* (p. 1)
>
> *Le calcul des probabilités offre une contradiction dans les termes qui servent à le désigner, et, si je ne craignais de rappeler ici un mot trop souvent répeté, je dirais qu'il nous enseigne surtout une chose: c'est de savoir que nous ne savons rien. Fin.* (p. 274)

The epistemological triangle

There is a double conceptual dependence that comes to the fore whenever the meaning of probability is at issue, as Hacking has expressed it: "[o]n the one side it is statistical, concerning itself with stochastic laws of chance processes. On the other side, it is epistemological, dedicated to assessing reasonable degrees of belief in propositions quite devoid of statistical background." (1975, p. 12). From a formal point of view, Bernoulli's theorem (the weak version of the law of large numbers) accounts for what probability is. This mathematical result can be used to express the duality of the concept of probability, since a gradually established statistical regularity is explained by the limiting *a priori* probability p which, in its turn, is explained by the tendency of the relative frequencies shown "in the long run". Nevertheless, it is this duality (an empirical/*a priori* idea) at the core of probability which seems to us to provide the best example of the way in which Steinbring has schematised the constitution of mathematical concepts (1997): as gradually built in from an interplay between contexts of reference (objects), symbols (signs) and the concepts themselves from previous stages in their constitution. He refers to this basic scheme in the constitution of mathematical concepts as the "epistemological triangle" (1997). With one example, Figure 1 freezes this spiral-like process, which evolves whenever the individual overcomes epistemological difficulties that force him to refine, modify or even to change his or her previous conceptual scheme. Figure 1 shows the way in which we identify the elements in the triangle with regard to the analysis we carry out in our researches. The example concerns the law of large numbers as it is proposed that secondary school teachers introduce it in the classroom setting. That is, to carry out sequences of Bernoulli's trials (for instance, tosses of a coin) and to register the occurrence of the outcome in each trial (heads or tails) in a diagram (sign) in order to focus on the gradually attained stability of the relative frequency (concept) (SEP, 1993). In the following section we explain how to register the outcomes.

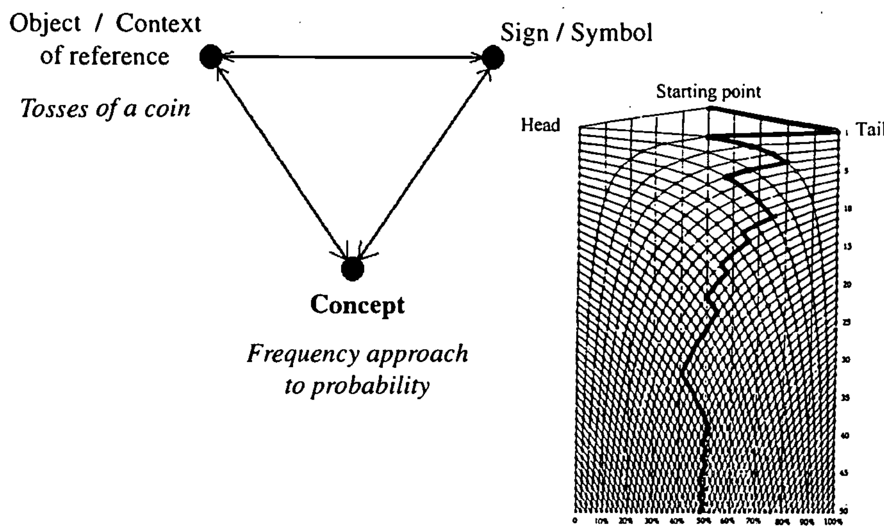

Figure 1. Example of the identification of the elements in the epistemological triangle.

The *object* toward which the individual's action is directed, or the *context* to which his or her activity *refers*, must be distinguished from the signs (symbols) used to denote the attribute or attributes at issue. Equally, in this scheme the object (or context of reference) must be distinguished from the concept. It is from the *reinterpretation* of the result obtained, or of the actions executed with regard to the object, -that is, to explain or to judge the result or the actions- that the constitution of the concept, or its evolution, takes place; concept formation builds on itself. This interpretation underlines the dynamic character of the process in the sense that it involves not only a sequence of actions to be reconsidered (reviewed) but their connection as well.

Three factors have to be pointed here. Firstly, that there is no restriction on the nature of the object, as its degree of abstraction can vary from physical world to conceptual objects, in particular, to the mathematical concepts themselves; that is, there is a transition from context-dependency to context-independent for the concept to evolve (Steinbring, 1998). Secondly, that it is the individual him or herself who is compelled to re-interpret his or her result and actions with regard to the object/context of reference in order for the concept to evolve. The individual's activity can refer to a mathematical concept, but the reinterpretation of the actions or of the result obtained regarding the object results either in its being refined, corrected or even discarded, depending on the degree of correspondence of previous experiences (with respect to that concept) and the reinterpretation made.

Thirdly, it seems necessary that notions be rooted by facing at first concrete physical situations from which a kind of ontological control could be established over the development of concepts, for the individual to make sense of his activity at every stage of abstraction. Notwithstanding this claim has been made on the grounds of theories of genetic epistemology (Piaget, 1951), which have greatly influenced curriculum design, it seems to be a postulate often neglected. In the case of probability, an example of overlooking this recommendation is the presentation of the classical definition for the *a priori* probability, as if this idea be innate, that is, as if it had emerged without any need of empirical acquaintance. Moreover, in the case of probability, it seems that contravening this order can result in difficulties in understanding the law of great numbers even at university level. For instance, even though the classical definition of probability follows from a logical reflection about the geometric properties of a physical random device (a die, coin, a pin, a spinner) considered as "ideal", taking it as the first step for teaching the law of large numbers instead of experiencing with the actual occurrences of possible outcomes from successions of independent Bernoulli's trials using one of such devices, can result in anchorage in the idea of equiprobability even when facing contrary evidence of relative frequency tendencies from long sequences of trials (De León, in progress with students of Social Sciences).

The epistemological triangle suggests that the evolving process of recurrence among object, sign and concept, results in the definition of a mathematical entity, which is more and more precise to the extent to which the individual has to reinterpret, to actively consider, his or her actions regarding the different aspects of the concept. However, this indication is not necessarily observed in probability education, and it is common instead to have the teaching of probability starting from definitions, even in open educational instances (e.g. see Vázquez for the case of Mexican TV secondary education, 1998).

It is worth stressing here the importance of the context of reference in the teaching of probability. Freudenthal pointed this out by stating that there should be awareness of the value of the isomorphism of problems (the same formal problem presented in different contexts) for the constitution of the mathematical entities. It is regarding the context of reference that psychological factors may give an account of drawbacks for the selection of the attributes from the context of reference concerning the concept. For instance, chronological order among events may be a drawback to understanding conditional probability (Ojeda, 1994; 1998).

We have referred to the epistemological triangle for the analysis of didactical activities on probability proposed for elementary and university mathematics education.

The role of semiotic registers of representation

In the epistemological triangle, the context of reference is as important as the signs we use to represent the attributes with which we are concerned and to have a physical support for establishing and keeping track of the logical relations among them.

Natural signs are constituted according to experience; they suggest to us the actions to be carried out. Hence, the interpretations that children make of the tasks they are asked about may not inform about their understanding of that task as was intended. Among other aspects, Gurrola (1998) studied the answers of children aged 6-7 years given to questions concerning the idea of chance, by proposing to them a seesaw trial to mix randomly equally sized marbles in two colours. The trial had a small divider in the middle as a reference for the arrangement of the marbles. The device was presented to a child, Almendra, at first showing all the marbles in one colour on one side, and all the marbles in the other colour on the other side. After several movements to and fro of the trial, it seemed that the girl did not have a notion of chance, which was in agreement with Piaget's results (1951). However, it turned out that she was focusing the task as a game to control the arrangement of the marbles since, pointing to the divider, she stated that *this is meant to keep them in their place*. Therefore, further questioning once the divider was removed revealed that she did have an idea of chance, what was confirmed with the explanation she gave using a drawing of the trajectories of the marbles that she did (see Figure 2).

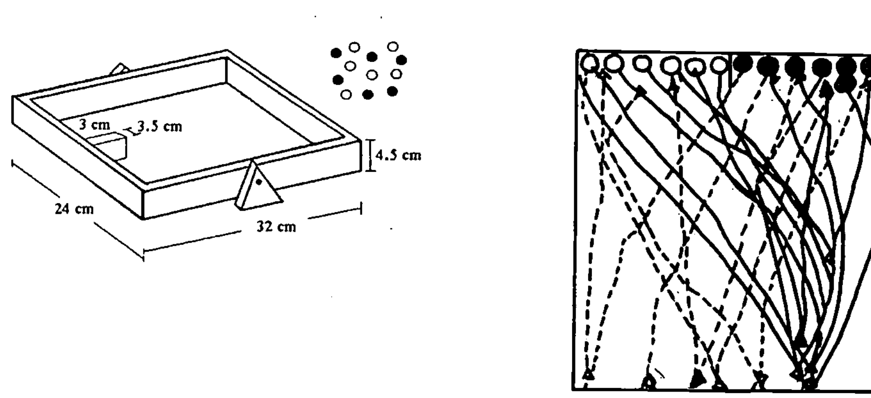

Figure 2. Figurative register produced from the random mixture experiment.

The signs we produce result from the need to overcome the limitations of natural signs (availability, being unwieldy) and to fix aspects of the context of reference, in order to focus on and to establish logical relations either among those attributes or with others. For secondary school pupils, labels for the relevant elements of a concrete physical situation (urns) had a mediator role between the situation and the diagrams that pupils produced to solve combinatorics problems referred to that situation (Elguea, 1998).

Written or printed forms are physical existences with a value of representative of meanings (of those attributes); they are produced under control and suggest the actions to be carried out in order to attain a target (to answer a question, to solve a doubt).

From the cognitive point of view, for the students to be introduced to the basic stages of the subject of probability, the use in teaching of means of organising the relevant information about a particular situation (context of reference) and to treat their data, results in prompting their mathematical activity (Ojeda, 1994). In more general terms, for the individual to develop and to communicate a mathematical activity, a system of signs, a semiotic register support, is necessary. A semiotic register, according to Duval (1996), constitutes a system of representation if it allows three cognitive fundamental activities: its production, an inside treatment, and an in-between treatment or conversion between different semiotic registers. The semiotic registers used in the mathematical activity are the algebraic, the graphical, the figurative and the natural language. Different aspects of a mathematical concept and of its levels of sophistication (formal structure) demand the staging of particular semiotic registers (Duval, 1996). For example, whereas the formal notation for Bernoulli's theorem (let $\varepsilon,\ \eta \in \Re$, ® $\varepsilon > 0$, ® $\eta > 0$, $\exists\, N \in \mathbf{N}$ ™ ® $n > N$, $P(|f_n - p| < \varepsilon) > 1 - \eta$) allows a more analytical and precise description of probability as a limit of relative frequencies, the diagram in Figure 3 shown some paragraphs below prefigures this result as an account of the frequency approach to probability for a particular random situation (a sequence of tosses of a coin).

Figurative registers provide a global structured organization of the relevant information for the students to have support to articulate their (mathematical) activity and to carry out a consecutive sequence of actions. For instance, at the same level of abstraction of conditional probability, the use of Venn diagrams with areas corresponding to the probabilities of the events represented, resulted in preparatory students understanding better the theorem of total probability, whereas tree diagrams favoured their understanding of Bayes' theorem (Barrera, 1994). Even though the work realized was complemented with the use of algebraic and numerical registers, the former figurative registers had a mediating role between the situation posed and the use of the mathematical results presented by means of formal notation, as the labels did in the case of secondary pupils.

However, the didactical value of the use of semiotic registers different from the algebraic register is not to be neglected with regard to the latter. More over, the semiotic registers cannot be used indiscriminately, without awareness of what they should be expected to suggest to the students, for their mathematical activity to be prompted. It is necessary to understand the way in which one can form and express with semiotic registers the specific mathematical result at issue vis-à-vis the cognizing individual. For instance, a tree diagram with proportionally varied scale so to allow the register of the outcomes from a relatively large number of trials of Bernoulli (50) is proposed in the guide for teachers (SEP, 1993) with the following instruction: *From the starting point, draw a line to the next point on the right if the outcome on the right occurs, and to the left if the outcome on the left occurs, and so on to the end* (p. 373). It consigns the number of trials on the right side and, at the bottom, the percentages corresponding to each branch, that is, the percentage of total occurrences of the outcome on the right, or path, in 50 trials. Figure 3 shows the resulting path using correctly the diagram for tosses of a coin, whereas Figure 4 was drawn by secondary school pupils during a classroom probability session, after the teacher's instructions (for each trial as quoted above), and according to his conducting of the class (Alquicira, 1998). The way in which this graph was drawn by the students prevented them from following the sequence in which the outcomes were occurring, as for each draw, the corresponding line started at the top of the graph; therefore, instead of one branch (path), there were as many branches as trials were carried out. Thus, no treatment of the register was possible. The diagram in Figure 4 does not allow one to thoroughly reconstruct the results of the sequence of trials; a precise reference to "number of occurrences" (as a random variable) is not possible there, as it can just be done by counting the "peaks" in Figure 3. Even more, an analytical way to obtain different paths ending in the same percentage, or in slightly different percentages, hence emphasising the idea of chance, cannot be worked out with Figure 4. The session in which this diagram was obtained was as follows. The teacher proposed that the class draws at random, with replacement, from an urn containing marbles in two colours, but in proportions unknown to the students:

> *We are going to carry out an experiment with the urn; you are going to mark on the graph* [he shows his own copy to the students] *with a line from the starting point, where the first point is in the*

first curve. Let's see; draw a line dividing at the middle [he points at an imaginary vertical line going down from the starting point on the top of the graph].

Thus, since the beginning, the class accepted (as they drew the line) what they were supposed to discover on the grounds of facts, that is, the proportions of the two kinds of marbles in the urn.

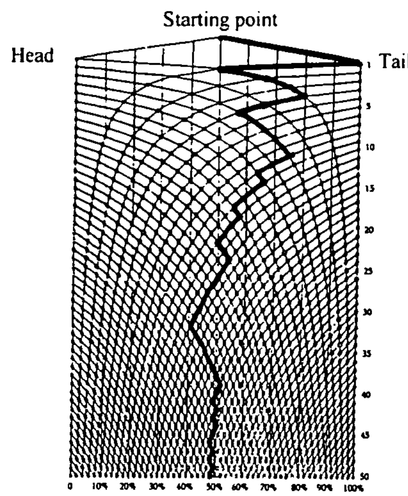

Figure 3. Tree diagram register sheet, produced correctly.

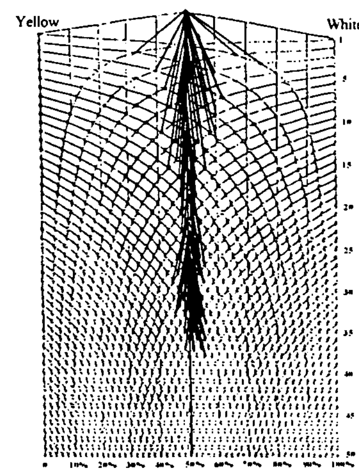

Figure 4. Pupil's graph after teacher's instructions.

On the left, if you agree, we are going to register the [occurrences of] *yellow marbles and on the other* [the occurrences of] *the white marbles. ... Let's carry out the experiment as many times as possible, and you are going to repeat* [say] *by watching the graph if there are more of these or of these* [he shows a marble of each colour], *and then we'll try to agree on how many of each colour. The graph may say if there are more of these or of these. And then we'll measure how many in all.*

After some draws were recorded, the teacher stated:

More we draw, more it's going to approach the corresponding probability, isn't it?

Having completed the sequence of 50 draws, the teacher said to the class:

> *This graph is pointing at 50%. Of course, that's probable, isn't it? It's not exact, it can't be exact ... We see there is a fluctuation, but it tends to 50%. The more repetitions we do, the more it's approaching 50%; that's a personal appreciation.*

At least three points may be considered here. Firstly, that the pupils were not given the opportunity to find in the tendency of the data a suggestion to relate the number of occurrences of the two possible outcomes in one trial with the *proportion* of the two kinds of marbles inside the urn, and to reflect about the conditions under which the phenomenon can be repeated; that is to say, to take the graph as a sign of the evidence obtained from the draws. Secondly, that transgressing the order by announcing "the" answer beforehand, deprived the activity of incentive for the students to make sense of the task proposed, to appropriate the question posed. Thirdly, this question was not answered at the end.

In terms of the epistemological triangle, the aim of the activity is misguided since the initial drawing of a line through the middle that the teacher proposed: using *his* knowledge of the proportions of the marbles in the urn (the context), he favoured the sign corner over the context of reference corner in the epistemological triangle. The class did not interpret the resulting 50% in terms of the context of reference (the composition in the urn). Finally, the teacher's remark at the end of the session regarding the progressive tendency towards 50% as to be taken as *personal appreciation* (opinion), withholds the activity from objectivity, whose search should be the aim when facing chance. As Hacking (1975) puts it, "[t]he old medieval probability was a matter of opinion ... [which] was probable if it was approved by ancient authority" (pp. 43-44).

Remarks

An account of the link among aspects of concepts (the elements, the relations they involve and their interpretations regarding different contexts), and their expression by means of semiotic registers can provide the research with grounds to carry out systematic analysis in order to complement the information concerning the understanding of those concepts.

The attention to registers produced by the individuals themselves involved in the research at issue is a way in which either their misinterpretations of the situation they are posed may be revealed or the information about their understanding of it may be complemented. However, research is also needed concerning the constitution and transition in the individual from natural signs to artificially produced signs (we refer here to written or printed signs).

Teacher training is needed on the use of semiotic registers in order to profit in probability education from the advantages that could be derived from didactical activities whose designing includes intentionally combining these resources on the basis of the aspects of the concepts at which they aimed. In the same vein, designers of textbooks and guides for teaching should be aware of the potential that the use of semiotic registers can supply to prefigure concepts of stochastics, and should provide instructions for that use accordingly.

References

Alquicira, M. A. (1998). *Probabilidad: Docencia y Praxis. Hacia una Fundamentación Epistemológica para la Educación Secundaria.* Unpublished Master Degree Thesis. Departamento de Matemática Educativa, Cinvestav-IPN, México.

Barrera, R. M. (1994). *Teorema de Bayes: Desarrollo Conceptual, Modelos de Enseñanza y Cognición*. Unpublished Master Degree Thesis. Departamento de Matemática Educativa, Cinvestav-IPN, México.

Chevallard, Y. (1991). *La Transposition Didactique: Du Savoir Savante au Savoir Enseigné.* La Pensée Sauvage, France.

De León, J. A. *Comprensión de la Idea de la Ley de los Grandes Números en Estudiantes del Nivel Superior*. (PhD. research project in progress). Departamento de Matemática Educativa, Cinvestav-IPN, México.

Duval, R. (1996). Quel Cognitif Retenir en Didactique des Mathématiques? *Recherches en Didactique des Mathématiques*. Vol. 16, No. 3, pp. 349-382.

Elguea, L. (1998). *Ideas de Combinatoria en el Alumno de Secundaria.* Unpublished Master Degree Thesis. Universidad Autónoma del Estado de Morelos, PNFAPM.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Reidel, Holland.

Galván, M. de la C. (1996). *Nubes y Relojes en la Curricula de Secundaria*. Unpublished Master Degree Thesis. Departamento de Matemática Educativa, Cinvestav-IPN, México.

González, E. (1995). *Estrategias de Niños de 3er. Grado de Primaria Frente a una Situación Lúdico-Aleatoria Experimental*. Unpublished Master Degree Thesis. Departamento de Matemática Educativa, Cinvestav-IPN, México.

Gurrola, M. L. (1998). *Pensamiento Probabilístico de Niños en Estadio Básico.* Unpublished Master Degree Thesis. Departamento de Matemática Educativa, Cinvestav-IPN, México.

Hacking, I. (1975). *The Emergence of Probability*. Cambridge University Press, Great Britain.

Heitele, D. (1975). An Epistemological View on Fundamental Stochastic Ideas. *Educational Studies in Mathematics*, *6*, pp. 187-205, Reidel, Holland.

Krüger, L. *et al.*, (eds.) (1987). *The Probabilistic Revolution*. MIT.

Limón, A. (1995). *Elementos para el Análisis Crítico de la Posible Inserción Curricular de Nociones de Estocásticos, Ausentes en el Programa de Preescolar y Primaria*. Unpublished Master's Degree Thesis. Departamento de Matemática Educativa, Cinvestav-IPN, México.

López, O. M. (1998). *Docencia y Enseñanza de Estocásticos en el Nivel Preescolar: El Recurso de la Representación*. Unpublished Master's Degree Thesis. Universidad Autónoma del Estado de Morelos, PNFAPM.

Merriam-Webster's Collegiate Dictionary.

Mosterin, J. (1964). *Conceptos y Teorías en la Ciencia*. (Ed.). Alianza. pp.11-39.

Ojeda, A. M. (1994). *Understanding Fundamental Ideas of Probability at Pre-University Level*. Unpublished Doctoral Thesis, King's College London, University of London.

Ojeda, A. M. (1998). Presentación de Problemas y Razonamiento Probabilístico. En 35 *Aniversario de Cinvestav del IPN. Investigaciones en Matemática Educativa* (Hitt, F., ed.). Grupo Editorial Iberoamérica.

Perrusquía, E. (1998). *Probabilidad y Aritmética: Estudio Epistemológico en el Estadio Medio: Dificultades de Interpretación*. Unpublished Master's Degree Thesis. Departamento de Matemática Educativa, Cinvestav-IPN, México.

Piaget, J.; Inhelder, B. (1951). *La Genèse de l'Idée de Hassard Chez l'Enfant*. PUF.

Poincaré, H. (1896). *Calcul des Probabilités. Leçons Professées Pendant le Deuxième Semestre*. A Quiquet, Paris.

Schmidt, S. (1996). Semantic Structures of Word Problems - Mediators Between Mathematical Structures and Cognitive Structures? In *8th International Congress on Mathematical Education. Selected Lectures*.(Alsina, C. *et al.* eds.), SAEM Thales, pp. 381-395.

S. E. P. (1993). *Libro para el maestro*. Matemáticas. Secundaria. México.

Sfard, A. (1996). On Acquisition Metaphor and Participation Metaphor for Mathematics Learning. In *8th International Congress on Mathematical Education. Selected Lectures*. Alsina, C. *et al.*, (eds.). SAEM Thales.

Steinbring, H. (1997). Epistemological Investigation of Classroom Interaction in Elementary Mathematics Teaching. *Educational Studies in Mathematics,* Vol. 32, pp. 49-92.

Steinbring, H. (1998). Epistemological Constraints of Mathematical Knowledge in Social Learning Settings. In *Mathematics Education as a Research Domain: A Search for Identity*. An ICMI Study, Book 2. Kluwer Academic Publishers, pp. 513-526.

Vázquez, G. (1998), *Enseñanza de la Probabilidad en Sistemas Abiertos: Identificación Teórica de Fundamentos. El Caso de la Telesecundaria en el Sistema Educativo Nacional.* Unpublished Master's Degree Thesis. Departamento de Matemática Educativa, Cinvestav-IPN, México.